2 The lock-in amplifier

The basic quantity to understand lock-in amplifier is Correlation Function

\[
R(\delta) = \frac{1}{T} \lim_{T \to \infty} \int_{0}^{T} f(t) g(t + \delta) \, dt
\]

If \( R \) at time \( \delta \) is zero, two functions are completely uncorrelated. If you don’t want to wait until infinity, the practical definition would include the time constant

\[
R(\delta, T) = \frac{1}{T} \int_{0}^{T} f(t) g(t + \delta) \, dt
\]

For locki, \( g(t) \) is the signal and \( f(t) \) is the reference. In the experiment

\[
f(t) = a \sin(\omega_0 t + \varphi)
\]

and

\[
g(t) = b \sin(\omega t + \Delta)
\]

(and can also contain harmonics). So, let’s say \( T \) is a period of frequency \( \omega_0 \)

\[
T_0 = \frac{2\pi}{\omega_0}
\]
and we can write an autocorrelation function

\[ R(\omega, \omega_0, n) = \frac{ab}{nT_0} \int_0^{nT_0} \sin(\omega t + \Delta) \sin(\omega_0 t + \varphi) \, dt \]

with

\[ 2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \]

Therefore,

\[ R(\omega, \omega_0, n) = \frac{ab}{2nT_0} \int_0^{nT_0} \left[ \cos([\omega - \omega_0] t + \Delta - \varphi) - \cos([\omega + \omega_0] t + \Delta + \varphi) \right] \, dt \]

The device (multiplier) has therefore two output signals - DC and \(2\omega_0\) (often square wave is used to cut all stuff). So at \(\omega = \omega_0\)

\[ R(\omega_0, n) = \frac{ab}{2nT_0} \left[ \cos(\Delta - \varphi) nT_0 \right] - \int_0^{nT_0} \cos(2\omega_0 t + \Delta + \varphi) \, dt \]

\[ \int_0^b \cos(x + A) \, dx = \sin(b + A) - \sin(A) \]
So with \( x = 2\omega_0 t \)

\[
\int_0^{nT_0} \cos (2\omega_0 t + \Delta + \varphi) \, dt = \frac{1}{2\omega_0} \int_0^{2\omega_0 nT_0} \cos (x + \Delta + \varphi) \, dx = \frac{1}{2\omega_0} [\sin (2\omega_0 nT_0 + \Delta + \varphi) - \sin (\Delta + \varphi)]
\]

but

\[
\sin (2\omega_0 nT_0 + \Delta + \varphi) = \sin (4\pi n + \Delta + \varphi) = \sin (\Delta + \varphi)
\]

and

\[
R(\omega_0) = \frac{ab}{2} \cos (\Delta - \varphi)
\]

Instrumentally, this is an INTERGRATOR that gets rid of the \( 2\omega_0 \) signal. Random noise also integrates out.

Great, but what about \( \Delta \)? Well, we have control over phase of the input. So, we can actually tune \( \varphi \) to maximize the signal and find condition at which

\[
\cos (\Delta - \varphi) = 1
\]

and

\[
R = \frac{ab}{2}
\]

Therefore, we measure both, the amplitude \( b \) and the phase shift at a given (by the reference) frequency \( \omega_0 \).
**CONCLUSION:** important parts of the lock-in are INTEGRATOR and Phase-sensitive Detector (PSD). And, of course multipliers. Importantly, we can use external references.

Imaginary representation (via Cotes (Euler) formula)

$$V = V_0 \exp(i\Delta) = V_0 \cos(\Delta) + iV_0 \sin(\Delta) = V' + iV''$$

Why do we need the lock-in? Sensitive measurements of only one component of the time-varying signal.
lock-in summary

\( g(t) \) is the signal and \( f(t) \) is the reference. In the experiment

\[
f(t) = a \sin(\omega_0 t + \varphi)
\]

and

\[
g(t) = b \sin(\omega t + \Delta)
\]

and we can write an autocorrelation function

\[
R(\omega, \omega_0, n) = \frac{ab}{nT_0} \int_0^{nT_0} \sin(\omega t + \Delta) \sin(\omega_0 t + \varphi) \, dt
\]

\[
R(\omega_0, n) = \frac{ab}{2nT_0} \left[ \cos(\Delta - \varphi) nT_0 \right] - \int_0^{nT_0} \cos(2\omega_0 t + \Delta + \varphi) \, dt
\]

and

\[
R(\omega_0) = \frac{ab}{2} \cos(\Delta - \varphi)
\]

Therefore, we measure both, the amplitude \( b \) and the phase shift at a given (by the reference) frequency \( \omega_0 \).
Basic principles of AC susceptometry

MULTIPLIER

PHASE SHIFT to MAX the signal – find the phase

Low-pass filter and integrator

DC out

SIGNAL

REF

Lock-in

Amplitude, Phase

Amplitude

Phase

Reference

Signal
AC magnetic susceptibility

\[ H = H_{dc} + H_{ac} \cos(\omega t) \]

\[ M(t) = M_0 + H_{ac} \sum_{n=1}^{\infty} \left[ \chi'_n(\omega) \cos(n\omega t) + \chi''_n(\omega) \sin(n\omega t) \right] \]

where

\[ \chi'_n(\omega) = \frac{\omega}{\pi H_{ac}} \int_{-\pi/\omega}^{\pi/\omega} M(t) \cos(n\omega t) \, dt \]

\[ \chi''_n(\omega) = \frac{\omega}{\pi H_{ac}} \int_{-\pi/\omega}^{\pi/\omega} M(t) \sin(n\omega t) \, dt \]

Note that \( n = 0 \) is special

\[ M_0 = \frac{1}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} M(t) \, dt \]

is simply an average (over period) value of magnetization. For example for

\[ M(t) = \text{const} = M_c \]

\[ M_0 = \frac{1}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} M_c \, dt = \frac{1}{2} M_c \omega \frac{2\pi}{\omega} = M_c \]
4 Losses per cycle.

Losses per cycle are determined by

\[ W = \int M(H) \, dH \]

or with

\[ H = H_{ac} \cos(\varphi) \]

\[ W = -H_{ac} \int_0^{2\pi} \left( M_0 + H_{ac} \sum_{n=1}^{\infty} \left[ \chi'_n(\omega) \cos(n\varphi) + \chi''_n(\omega) \sin(n\varphi) \right] \right) \sin(\varphi) \, d\varphi \]

\[ \int_0^{2\pi} \cos(n\varphi) \sin(\varphi) \, d\varphi = 0 \]

\[ \int_0^{2\pi} \sin(\varphi) \sin(\varphi) \, d\varphi = \pi \]

\[ \int_0^{2\pi} \sin(n\varphi) \sin(\varphi) \, d\varphi = 0, \quad n \neq 1 \]

So,

\[ W = \pi H_{ac}^2 \chi_1 \]
typical AC susceptometer
In ac experiment $\bar{B}$ is a function of time and controls the voltage in one turn of the pick-up coil:

$$V_m(t) = -\frac{\partial \Phi_m(t)}{\partial t} = -A \frac{\partial \bar{B}(t)}{\partial t}$$  \hspace{1cm} (7)

It is important to know that the pick-up coil measures an integrated value of the flux density in the sample. Using eqn. (6) one can define the sample magnetization as:

$$M(t) = \bar{B}(t) - B_{\text{ext}}(t) = \frac{\Phi_m(t)}{A} - B_{\text{ext}}(t)$$  \hspace{1cm} (8)
Exercise 3 (a) For a solenoid of radius $R$ and length $D \gg R$ carrying a current $I_s$, the self-induced flux through the solenoid is defined by $\Phi_s = NB_sA$, where $N = nD$ is the total number of turns and $A = \pi R^2$ is the cross-sectional area. Show that $\Phi_s$ can be expressed

$$\Phi_s = LI_s$$

where $L = \mu_0 n^2 V$, and $V$ is the solenoid volume.

$L$ is called the self inductance.
now we place it in field

The solenoid is placed with its axis along an applied field $B_a$.

**Exercise 4** Show that the flux through the solenoid due to the applied field $\Phi = NB_aA$ can be written

$$\Phi_a = LI_a$$  \hspace{1cm} (13)

where the pseudo-current $I_a$ is defined by $I_a = B_a/\mu_0 n$.

Thus, if the solenoid carries a current $I_s$, the net flux through the solenoid can be expressed

$$\Phi = L(I_s + I_a)$$  \hspace{1cm} (14)
Exercise 5 Show that if $B_a$ is given by Eq. 4 (i.e., $I_a = I_{a0} \cos \omega t$ with $I_{a0} = H_0/n$), then the current $I_s$ in the solenoid will be

$$I_s = I_{a0}(\chi' \cos \omega t + \chi'' \sin \omega t)$$

where

$$\chi' = -\frac{\omega^2 L^2}{R^2 + \omega^2 L^2}$$

$$\chi'' = \frac{R \omega L}{R^2 + \omega^2 L^2}$$
collective behavior (spin glass, vortices, superparamagnetic particles)

Figure 1. AC susceptibility of CuMn (1 at% Mn) showing the cusp at the freezing temperature. The inset shows the frequency dependence of the cusp from 2.6 Hz (triangles) to 1.33 kHz (squares). Figure reprinted with permission.²
Figure 3. AC susceptibility of $\text{LaBaCa(Cu}_{1-x}\text{Zn}_x\text{)O}_{7.8}$ for various concentrations of Zn. From the real part of the susceptibility, the authors determined the critical temperature. By examining the peak location and the width of the peaks in the imaginary susceptibility, the authors were able to understand the behavior of the superconductivity in the weak links between grains in the sample. Figure reprinted with permission.\textsuperscript{11}
local AC response
even simpler device – measure $B(x)$

Local ac magnetic response in type-II superconductors

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FIG. 1. Schematic description of the magnetic induction during one cycle. Straight lines are only for simplicity of the plot.
FIG. 2. Curves of $B_z(\varphi)$ as calculated from Table I; for $H^* = 0$ and $H^* = H_{ac}/2$. 

the wave forms
TABLE I. Magnetic induction $B_z(\varphi)$ at the Hall-probe location during one cycle of the ac field. The parameter $x$ denotes the ratio $H^*/H_{ac}$.

<table>
<thead>
<tr>
<th>Stage No. (see Fig. 1)</th>
<th>$\varphi$</th>
<th>$B_z(\varphi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 2</td>
<td>$0 \rightarrow \frac{\pi}{2}$</td>
<td>$H_{ac} \sin(\varphi) - H^*$</td>
</tr>
<tr>
<td>2 to 3</td>
<td>$\frac{\pi}{2} \rightarrow \pi + \arcsin(2x - 1)$</td>
<td>$H_{ac} - H^*$</td>
</tr>
<tr>
<td>3 to 4</td>
<td>$\pi + \arcsin(2x - 1) \rightarrow \frac{3\pi}{2}$</td>
<td>$H_{ac} \sin(\varphi) + H^*$</td>
</tr>
<tr>
<td>4 to 5</td>
<td>$\frac{3\pi}{2} \rightarrow 2\pi + \arcsin(2x - 1)$</td>
<td>$H^* - H_{ac}$</td>
</tr>
<tr>
<td>5 to 6</td>
<td>$2\pi + \arcsin(2x - 1) \rightarrow 2\pi$</td>
<td>$H_{ac} \sin(\varphi) - H^*$</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>$\chi'_1$</th>
<th>$\frac{\pi}{2} - 2\rho(2x-1) - \arcsin(2x-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi'_1$</td>
<td>$-4\rho^2$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$\sqrt{\frac{\pi^2}{4} + 2(2x-1)\rho[2 \arcsin(2x-1) - \pi] + 4\rho^2}$</td>
</tr>
<tr>
<td></td>
<td>$+ \arcsin(2x-1)[\arcsin(2x-1) - \pi]$</td>
</tr>
<tr>
<td>$\chi'_3$</td>
<td>$\frac{16}{3} \rho^3(2x-1)$</td>
</tr>
<tr>
<td>$\chi'_5$</td>
<td>$\frac{4}{3} \rho^2(1 - 8\rho^2)$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$\frac{4}{3} \rho^2$</td>
</tr>
<tr>
<td></td>
<td>The maximum $A_3 = \frac{1}{3}$ is reached at $x = \rho = \frac{1}{2}$.</td>
</tr>
<tr>
<td>$\chi'_3$</td>
<td>$\frac{16}{15} \rho^3(2x-1)(5 - 32\rho^2)$</td>
</tr>
<tr>
<td>$\chi'_5$</td>
<td>$\frac{4}{15} \rho^2(8\rho^2(9 - 32\rho^2) - 3)$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$\frac{4}{15} \rho^2\sqrt{9 - 32\rho^2}$</td>
</tr>
<tr>
<td></td>
<td>$A_5$ has two maxima: $A_5 = \frac{\sqrt{3}}{20}$ at $x = \frac{1}{4}$ and $\frac{3}{4}$, and a minimum $A_5 = \frac{1}{15}$ at $x = \frac{1}{2}$.</td>
</tr>
</tbody>
</table>
FIG. 3. Third and fifth harmonic signals measured in YBCO thin film as function of $H_{ap}$ at $T=88$ K and $H_{dc}=200$ G. Symbols are experimental data and solid lines are fits.
Frequency dependence of the local ac magnetic response in type-II superconductors

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The local ac magnetic response in type-II superconductors is analyzed on the basis of the critical state model, taking into account magnetic relaxation effects. The results show that the frequency must be introduced as an independent parameter to the model, in addition to the shielding current which itself is a function of frequency. Using a simplified model for the relaxation law, the calculated frequency dependence of the third harmonic response compares well with experimental data obtained in a Y-Ba-Cu-O crystal. Application of this analysis to ac measurements of magnetic relaxation in the short time limit is discussed.
example of frequency dependence

FIG. 1. Temperature dependence of the third harmonic response $V_3(T)$ in Y-Ba-Cu-O crystal for the indicated frequencies [after Wolfus et al. (Ref. 10)]. Note the pronounced frequency dependence of the peak height and the width.
The influence of vortex creep

Activation energy behavior

\[ F_L = \Phi_0 J \]

Pinning

- Vacancies, voids, inhomogeneities, where superconductivity is weak
- Pinning decreases energy losses caused by flux creep
FIG. 2. Schematic description of magnetic induction profiles during one cycle of the applied ac field. A miniature Hall probe is located at a distance $r$ from the edge of the sample ($r=0$). The profiles 1–6 refer to different times during the ac period; see text.
and how it affects the wave-forms

FIG. 3. Wave forms of the magnetic induction during one cycle, for $x = H^*/H_{ac} = 0.5$, with ($\omega t_0 = 5$) and without relaxation (solid and dotted lines, respectively). Numbers correspond to the stages in Fig. 2.
local vs. global AC susceptibility

Once the wave form of $B_z$ is known, the real part $\chi'_n$, imaginary part $\chi''_n$, and magnitude $A_n = \sqrt{(\chi'_n)^2 + (\chi''_n)^2}$ of the local harmonic susceptibilities can be calculated:

$$\begin{cases} \chi'_n \\ \chi''_n \end{cases} = \frac{1}{H_{ac}} \int_0^{2\pi} B(\varphi) \begin{bmatrix} \sin(n\varphi) \\ \cos(n\varphi) \end{bmatrix} d\varphi \quad (7)$$

(note that in the normal state our definition gives $\chi'_1 = \pi$). The global harmonic susceptibilities of the entire sample can be calculated by integration:

$$X_n = \frac{\delta}{\pi} \int_0^k \chi_n(x) dx, \quad (8)$$

where $\delta = H_{ac}/H_p$ is the parameter of the global response defined in Ref. 12, (inversely) analogous to our local parameter $x$, and $H_p$ is the field of full penetration up to the center of a sample. The upper integration limit

$$k = \begin{cases} 1/\delta & \text{if } \delta \geq 1 \\ 1 & \text{if } \delta \leq 1 \end{cases} \quad (9)$$
AC susceptometers

- A true AC susceptometer must have an AC component of the applied field.
- The use of lock-in amplifier does not guarantee that the device is an AC susceptometer.

\[
V_{\text{Hall}} = R_H I_{\text{DC}} H_{\text{DC}}
\]

\[
V_{\text{Hall}} = R_H I_{\text{AC}} H_{\text{DC}}
\]
different types and designs – the amplitude domain
Fig. 1. Schematic of the L/S sample zone, not drawn to scale. For clarity, Kel-F plastic springs are not shown.
QD AC coil set for PPMS

AC Susceptibility Sensitivity: \(2 \times 10^{-8} \text{ emu} \) (\(2 \times 10^{-11} \text{ Am}^2\)) @ 10 kHz
Analog vs frequency-domain measurements

advantages of the frequency domain
- arbitrary wave form
- bandpass filtering
- mixing
- aggressive amplification
- extremely stable standards
 resonant techniques

\[ Z = Z_L + Z_C \]

By writing the inductive impedance as \( Z_L = j\omega L \) and capacitive impedance as \( Z_C = \frac{1}{j\omega C} \) and substituting we have

\[ Z = j\omega L + \frac{1}{j\omega C}. \]

Writing this expression under a common denominator gives

\[ Z = \frac{(\omega^2 LC - 1)j}{\omega C}. \]

Hence, at \( f_r \):

\[ X_L = -X_C \]

\[ \omega L = \frac{1}{\omega C} \]

Converting angular frequency into hertz we get

\[ 2\pi f L = \frac{1}{2\pi f C} \]

Here \( f \) is the resonant frequency. Then rearranging,

\[ f = \frac{1}{2\pi \sqrt{LC}} \]

measure resonant frequency SHIFT!

- Below \( f_r \), \( X_L \ll (-X_C) \). Hence circuit is capacitive.
- Above \( f_r \), \( X_L \gg (-X_C) \). Hence circuit is inductive.
driven vs. self-resonating circuit

problems: phase noise and finite Q - factor

self-resonating circuit is equivalent to an infinite - Q resonator. phase noise is the only issue (can be dealt with with ultra-high stability clocks)
Figure 1. Schematic diagram of the microwave system used for measurements on the coaxial cavity system.
microwave cavity-perturbation technique

Electromagnetic field configurations in a TE\(_{102}\) mode rectangular resonant cavity of dimensions \(a\), \(b\), and \(d\).

Diagramatic sketch of the TE\(_{012}\) (left) and TE\(_{112}\) (right) cylindrical resonant cavity modes.
what is measured and the calibration

\[ \Delta \left( \frac{1}{2Q} \right) = \frac{R_S}{G}, \quad (1a) \]

\[ \frac{\Delta f}{f} = \frac{X_S - C}{G}, \quad (1b) \]

where \( G \) is the geometrical factor and \( C \) is the metallic shift which is the frequency shift caused by the insertion of a perfect conductor with the same volume as the sample. This relation can be understood physically as follows. The \( Q \) represents the degree of the energy dissipation and \( f \) represents the effective space that microwave electromagnetic field can spread out. Indeed, \( X_S = \mu \omega \delta / 2 \) for a normal metal, and \( X_S = \mu \omega \lambda \) for a superconductor, where \( \delta \) and \( \lambda \) are the skin depth and the penetration depth of the sample, respectively, \( \omega \) is the angular frequency, and \( \mu \) is the permeability of the sample. For materials of interest in this paper, \( \mu \) can be regarded as the permeability of vacuum, \( \mu_0 \). In the analysis of the experimental data, \( G \) and \( C \) are usually considered to be constants, and determined by the DC resistivity, \( \rho_{DC} \), utilizing the relation,

\[ R_S = X_S = \left( \mu_0 \omega \rho_{DC} / 2 \right)^{1/2}. \quad (2) \]