Magnetic field

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Physics 590B
popular definitions

• **Magnetic fields** surround magnetic materials and electric currents and are detected by the force they exert on other magnetic materials and moving electric charges. The magnetic field at any given point is specified by both a direction and a magnitude (or strength)

• **magnetized region of space**: a region of space surrounding a magnetized body or current-carrying circuit in which the resulting magnetic force can be detected

• **A condition** found in the region around a magnet or an electric current, characterized by the existence of a detectable magnetic force at every point in the region and by the existence of magnetic poles.

• **A magnetic field** is generated when electric charge carriers such as electrons move through space or within an electrical conductor.
Earliest description of a magnetic field was performed by Petrus Peregrinus and published in his “Epistola Petri Peregrini de Maricourt ad Sygerum de Foucaucourt Militem de Magnete” and is dated 1269 A.D. Petrus Peregrinus mapped out the magnetic field on the surface of a spherical magnet. Noting that the resulting field lines crossed at two points he named those points 'poles' in analogy to Earth's poles.

Three centuries later, near the end of the sixteenth century, William Gilbert of Colchester replicated Petrus Peregrinus work and was the first to state explicitly that Earth itself was a magnet. William Gilbert's great work De Magnete was published in 1600 A.D. and helped to establish the study of magnetism as a science.

The modern distinction between the B- and H- fields was not needed until Siméon-Denis Poisson (1781–1840) developed one of the first mathematical theories of magnetism. Poisson's model, developed in 1824, assumed that magnetism was due to magnetic charges. In analogy to electric charges, magnetic charges produce an H-field. In modern notation, Poisson's model is exactly analogous to electrostatics with the H-field replacing the electric field E-field and the B-field replacing the auxiliary D-field.

Hans Christian Oersted discovered that an electrical current generates a magnetic field that encircles the wire.

Andre Marie Ampere showed that parallel wires having currents in the same direction attract

Jean-Baptiste Biot and Felix Savart developing the correct equation, the Biot-Savart Law, for the magnetic field of a current carrying wire.

In 1825, Ampere published his Ampere's Law which provided a more mathematically subtle and correct description of the magnetic field generated by a current than the Biot-Savart Law.

Michael Faraday shows in 1831 that a changing magnetic field generates an encircling electric field.

In 1861, James Clerk-Maxwell wrote a paper entitled 'On Physical Lines of Force' in which he attempted to explain Faraday's magnetic lines of force in terms of a sea of tiny molecular vortices. These molecular vortices occupied all space and they were aligned in a solenoidal fashion such that their rotation axes traced out the magnetic lines of force.

Although the classical theory of electrodynamics was essentially complete with Maxwell's equations, the twentieth century saw a number of improvements and extensions to the theory. Albert Einstein, in his great paper of 1905 that established relativity, showed that both the electric and magnetic fields were part of the same phenomena viewed from different reference frames. Finally, the emergent field of quantum mechanics was merged with electrodynamics to form quantum electrodynamics or QED.
A sketch of Earth's magnetic field representing the source of Earth's magnetic field as a magnet. The north pole of earth is near the top of the diagram, the south pole near the bottom. Notice that the south pole of that magnet is deep in Earth's interior below Earth's North Magnetic Pole. Earth's magnetic field is produced in the outer liquid part of its core due to a dynamo that produce electrical currents there.
magnetic field due to current

cgs

- $H$ (Oe)
- $B$ (G)

SI

- $H$ (A/m)
- $B$ (T)
the right-hand rule
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning (first term is the most common)</th>
<th>SI Unit of Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>electric field</td>
<td>volt per meter or, equivalently, newton per coulomb</td>
</tr>
<tr>
<td>B</td>
<td>magnetic field</td>
<td>tesla, or equivalently, weber per square meter or, equivalently, volt-second per square meter</td>
</tr>
<tr>
<td></td>
<td>also called the magnetic induction</td>
<td></td>
</tr>
<tr>
<td></td>
<td>also called the magnetic field density</td>
<td></td>
</tr>
<tr>
<td></td>
<td>also called the magnetic flux density</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>electric displacement field</td>
<td>coulombs per square meter or, equivalently, newton per volt-meter</td>
</tr>
<tr>
<td></td>
<td>also called the electric flux density</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>magnetizing field</td>
<td>ampere per meter</td>
</tr>
<tr>
<td></td>
<td>also called auxiliary magnetic field</td>
<td></td>
</tr>
<tr>
<td></td>
<td>also called magnetic field intensity</td>
<td></td>
</tr>
<tr>
<td></td>
<td>also called magnetic field</td>
<td></td>
</tr>
<tr>
<td>( \nabla \cdot )</td>
<td>the divergence operator</td>
<td>per meter (factor contributed by applying either operator)</td>
</tr>
<tr>
<td>( \nabla \times )</td>
<td>the curl operator</td>
<td>per second (factor contributed by applying the operator)</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial t} )</td>
<td>partial derivative with respect to time</td>
<td></td>
</tr>
<tr>
<td>dA</td>
<td>differential vector element of surface area ( A ), with infinitesimally small magnitude and direction normal to surface ( S )</td>
<td>square meters</td>
</tr>
<tr>
<td>dl</td>
<td>differential vector element of path length tangential to the path/curve</td>
<td>meters</td>
</tr>
<tr>
<td>( \varepsilon_0 )</td>
<td>permittivity of free space, officially the electric constant, a universal constant</td>
<td>farads per meter</td>
</tr>
<tr>
<td>( \mu_0 )</td>
<td>permeability of free space, officially the magnetic constant, a universal constant</td>
<td>henries per meter or, Newtons per ampere squared</td>
</tr>
<tr>
<td>( \rho_f )</td>
<td>free charge density (not including bound charge)</td>
<td>coulombs per cubic meter</td>
</tr>
<tr>
<td>( \rho )</td>
<td>total charge density (including both free and bound charge)</td>
<td>coulombs per cubic meter</td>
</tr>
<tr>
<td>( \mathbf{J}_f )</td>
<td>free current density (not including bound current)</td>
<td>amperes per square meter</td>
</tr>
<tr>
<td>( \mathbf{J} )</td>
<td>total current density (including both free and bound current)</td>
<td>amperes per square meter</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Unit</td>
</tr>
<tr>
<td>--------</td>
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<td>------</td>
</tr>
<tr>
<td>$Q_f(V)$</td>
<td>net unbalanced free electric charge in the interior of an arbitrary closed surface $S = \partial V$ (not including bound charge)</td>
<td>coulombs</td>
</tr>
<tr>
<td>$Q(V)$</td>
<td>net unbalanced electric charge in the interior of an arbitrary closed surface $S = \partial V$ (including both free and bound charge)</td>
<td>coulombs</td>
</tr>
<tr>
<td>$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{l}$</td>
<td>line integral of the electric field along the boundary $\partial S$ (therefore necessarily a closed curve) of the surface $S$</td>
<td>joules per coulomb</td>
</tr>
<tr>
<td>$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{l}$</td>
<td>line integral of the magnetic field over the closed boundary $\partial S$ of the surface $S$</td>
<td>tesla-meters</td>
</tr>
<tr>
<td>$\iint_{\partial V} \mathbf{E} \cdot d\mathbf{A}$</td>
<td>the flux of the electric field through any closed surface $S = \partial V$</td>
<td>joule-meter per coulomb</td>
</tr>
<tr>
<td>$\iint_{\partial V} \mathbf{B} \cdot d\mathbf{A}$</td>
<td>the flux of the magnetic field through any closed surface $S = \partial V$</td>
<td>tesla meters-squared or webers</td>
</tr>
<tr>
<td>$\iint_{S} \mathbf{B} \cdot d\mathbf{A} = \Phi_{B,S}$</td>
<td>magnetic flux through any surface $S$ (not necessarily closed)</td>
<td>webers or equivalently, volt-seconds</td>
</tr>
<tr>
<td>$\iint_{S} \mathbf{E} \cdot d\mathbf{A} = \Phi_{E,S}$</td>
<td>electric flux through any surface $S$, not necessarily closed</td>
<td>joule-meters per coulomb</td>
</tr>
<tr>
<td>$\iint_{S} \mathbf{D} \cdot d\mathbf{A} = \Phi_{D,S}$</td>
<td>flux of electric displacement field through any surface $S$, not necessarily closed</td>
<td>coulombs</td>
</tr>
<tr>
<td>$\iint_{S} \mathbf{J}<em>f \cdot d\mathbf{A} = I</em>{f,s}$</td>
<td>net free electrical current passing through the surface $S$ (not including bound current)</td>
<td>amperes</td>
</tr>
<tr>
<td>$\iint_{S} \mathbf{J} \cdot d\mathbf{A} = I_S$</td>
<td>net electrical current passing through the surface $S$ (including both free and bound current)</td>
<td>amperes</td>
</tr>
</tbody>
</table>
### The Maxwell's Equations

**Formulation in terms of free charge and current**

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential form</th>
<th>Integral form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss's law</td>
<td>$\nabla \cdot D = \rho_f$</td>
<td>$\iiint D \cdot dA = Q_f(V)$</td>
</tr>
<tr>
<td>Gauss's law for magnetism</td>
<td>$\nabla \cdot B = 0$</td>
<td>$\iiint B \cdot dA = 0$</td>
</tr>
<tr>
<td>Maxwell–Faraday equation (Faraday's law of induction)</td>
<td>$\nabla \times E = -\frac{\partial B}{\partial t}$</td>
<td>$\oint E \cdot dl = -\frac{\partial \Phi_{B,S}}{\partial t}$</td>
</tr>
<tr>
<td>Ampère's circuital law (with Maxwell's correction)</td>
<td>$\nabla \times H = J_f + \frac{\partial D}{\partial t}$</td>
<td>$\oint H \cdot dl = I_{f,S} + \frac{\partial \Phi_{D,S}}{\partial t}$</td>
</tr>
<tr>
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<td>-------------------------------------------</td>
<td>-----------------------------------------------------------------------------------</td>
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</tr>
<tr>
<td>Gauss's law</td>
<td>$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$</td>
<td>$\iiint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{Q(V)}{\varepsilon_0}$</td>
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</tr>
<tr>
<td>Ampère's circuital law (with Maxwell's correction)</td>
<td>$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$</td>
<td>$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t}$</td>
</tr>
</tbody>
</table>
in cgs units

Gaussian units, the equations take the following form:

\[ \nabla \cdot \mathbf{D} = 4\pi \rho_f \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{J}_f \]

where \( c \) is the speed of light in a vacuum. For the electromagnetic field in a vacuum, the equations become:

\[ \nabla \cdot \mathbf{E} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \]

In this system of units the relation between displacement field, electric field and polarization density is:

\[ \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \]

And likewise the relation between magnetic induction, magnetic field and total magnetization is:

\[ \mathbf{B} = \mathbf{H} + 4\pi \mathbf{M} \]

In the linear approximation, the electric susceptibility and magnetic susceptibility can be defined so that:

\[ \mathbf{P} = \chi_e \mathbf{E}, \quad \mathbf{M} = \chi_m \mathbf{H} \]

(Nota that although the susceptibilities are dimensionless numbers in both cgs and SI, they have different values in the two unit systems, by a factor of \( 4\pi \).) The permittivity and permeability are:

\[ \epsilon = 1 + 4\pi \chi_e, \quad \mu = 1 + 4\pi \chi_m \]

so that:

\[ \mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H} \]

In vacuum, one has the simple relations \( \epsilon = \mu = 1 \), \( \mathbf{D} = \mathbf{E} \), and \( \mathbf{B} = \mathbf{H} \)

The force exerted upon a charged particle by the electric field and magnetic field is given by the Lorentz force equation:

\[ \mathbf{F} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \]

where \( q \) is the charge on the particle and \( \mathbf{v} \) is the particle velocity. This is slightly different from the SI-unit expression above. For example, here the magnetic field \( \mathbf{B} \) has the same units as the electric field \( \mathbf{E} \).

Some equations in the article are given in Gaussian units, but not SI or vice-versa. Fortunately, there are general rules to convert from one to the other; see the article Gaussian units for details.
original Maxwell’s equations

Maxwell’s A Dynamical Theory of the Electromagnetic Field (1864)

In 1864 Maxwell published A Dynamical Theory of the Electromagnetic Field in which he showed that light was an electromagnetic phenomenon. Confusion over the term “Maxwell’s equations” is exacerbated because it is also sometimes used for a set of eight equations that appeared in Part II of Maxwell’s 1864 paper A Dynamical Theory of the Electromagnetic Field, entitled “General Equations of the Electromagnetic Field”,[3] a confusion compounded by the writing of six of those eight equations as three separate equations (one for each of the Cartesian axes), resulting in twenty equations in twenty unknowns. (As noted above, this terminology is not common. Modern references to the term “Maxwell’s equations” refer to the Heaviside re-statement.)

The eight original Maxwell’s equations can be written in modern vector notation as follows:

(A) The law of total currents

\[ \mathbf{J}_{\text{tot}} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

(B) The equation of magnetic force

\[ \mu \mathbf{H} = \nabla \times \mathbf{A} \]

(C) Ampère’s circuit law

\[ \nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t \]

(D) Electromotive force created by convection, induction, and by static electricity. (This is in effect the Lorentz force)

\[ \mathbf{E} = \mu \nabla \times \mathbf{H} - \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \]

(E) The electric elasticity equation

\[ \mathbf{E} = \frac{1}{\varepsilon} \mathbf{D} \]

(F) Ohm’s law

\[ \mathbf{E} = \frac{1}{\sigma} \mathbf{J} \]

(G) Gauss’s law

\[ \nabla \cdot \mathbf{D} = \rho \]

(H) Equation of continuity

\[ \nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t} \]

Notation:

- \( \mathbf{H} \) is the magnetizing field, which Maxwell called the “magnetic intensity”.
- \( \mathbf{J} \) is the electric current density (with \( \mathbf{J}_{\text{tot}} \) being the total current including displacement current).
- \( \mathbf{D} \) is the displacement field (called the “electric displacement” by Maxwell).
- \( \rho \) is the free charge density (called the “quantity of free electricity” by Maxwell).
- \( \mathbf{A} \) is the magnetic vector potential (called the “angular impulse” by Maxwell).
- \( \mathbf{E} \) is called the “electromotive force” by Maxwell. The term electromotive force is nowadays used for voltage, but it is clear from the context that Maxwell’s meaning corresponded more to the modern term electric field.
- \( \phi \) is the electric potential (which Maxwell also called “electric potential”).
- \( \sigma \) is the electric conductivity (which Maxwell called the inverse of conductivity the “specific resistance”, what is now called the resistivity).

It is interesting to note the \( \mu \nabla \times \mathbf{H} \) term that appears in equation D. Equation D is therefore effectively the Lorentz force, similarly to equation (77) of his 1861 paper (see above).

When Maxwell derives the electromagnetic wave equation in his 1865 paper, he uses equation D to cater for electromagnetic induction rather than Faraday’s law of induction which is used in modern textbooks. (Faraday’s law itself does not appear among his equations.) However, Maxwell drops the \( \mu \nabla \times \mathbf{H} \) term from equation D when he is deriving the electromagnetic wave equation, as he considers the situation only from the rest frame.
### magnetic monopoles

<table>
<thead>
<tr>
<th>Name</th>
<th>Without magnetic monopoles</th>
<th>With magnetic monopoles (hypothetical)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law:</td>
<td>$\mathbf{\nabla} \cdot \vec{E} = 4\pi \rho_e$</td>
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<td>Gauss’s law for magnetism:</td>
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<td>Maxwell–Faraday equation (Faraday’s law of induction):</td>
<td>$-\mathbf{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$</td>
<td>$-\mathbf{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} + 4\pi \vec{j}_m$</td>
</tr>
<tr>
<td>Ampère’s law (with Maxwell’s extension):</td>
<td>$\mathbf{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{j}_e$</td>
<td>$\mathbf{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{j}_e$</td>
</tr>
</tbody>
</table>

Note: the Bivector notation embodies the sign swap, and these four equations can be written as only one equation.
magnetic vector potential

The magnetic vector potential \( \mathbf{A} \) is a three-dimensional vector field whose curl is the magnetic field, i.e.:

\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]

Since the magnetic field is divergence-free (i.e. \( \nabla \cdot \mathbf{B} = 0 \), called Gauss’s law for magnetism), this guarantees that \( \mathbf{A} \) always exists (by Helmholtz’s theorem).

Unlike the magnetic field, the electric field is derived from both the scalar and vector potentials:

\[
\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}.
\]

Starting with the above definitions:

\[
\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0
\]

\[
\nabla \times \mathbf{E} = \nabla \times \left( -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right) - \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) - \frac{\partial \mathbf{B}}{\partial t}.
\]

Note that the divergence of a curl will always give zero. Conveniently, this solves the second and third of Maxwell’s equations automatically, which is to say that a continuous magnetic vector potential field is guaranteed not to result in magnetic monopoles.

The vector potential \( \mathbf{A} \) is used when studying the Lagrangian in classical mechanics and in quantum mechanics (see Schrödinger equation for charged particles, Dirac equation, Aharonov-Bohm effect).

In the SI system, the units of \( \mathbf{A} \) are volt seconds per metre (V \cdot s \cdot m^{-1}).

Gauge choices

Main article: Gauge fixing

It should be noted that the above definition does not define the magnetic vector potential uniquely because, by definition, we can arbitrarily add curl-free components to the magnetic potential without changing the observed magnetic field. Thus, there is a degree of freedom available when choosing \( \mathbf{A} \). This condition is known as gauge invariance.

Magnetic scalar potential

The magnetic scalar potential is another useful tool in describing the magnetic field around a current source. It is only defined in regions of space in the absence of currents.

The magnetic scalar potential is defined by the equation:

\[
\mathbf{B} = -\mu_0 \nabla \psi.
\]

Applying Ampère’s law to the above definition we get:

\[
\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} = -\nabla \times \nabla \psi = 0.
\]

Solenoidality of the magnetic field leads to Laplace’s equation for potential:

\[
\Delta \psi = 0.
\]

Since in any continuous field, the curl of a gradient is zero, this would suggest that magnetic scalar potential fields cannot support any sources. In fact, sources can be supported by applying discontinuities to the potential field (thus the same point can have two values for points along the discontinuity). These discontinuities are also known as “cuts”. When solving magnetostatics problems using magnetic scalar potential, the source currents must be applied at the discontinuity.
magnetic field momentum

Field momentum
Moving charge experiences "friction" off the magnetic field. Although $B$ exerts only perpendicular Lorentz force. This field momentum is obtained by integrating the Poynting vector,

$$ p_{\text{field}} = \frac{1}{4\pi} \int dV E \times B $$

in non-relativistic case we can think of $B$ only due to external sources and $E$ comes from the charge. With electrostatic potential $\varphi$ for charge $q$ at a position $r'$

$$ E = -\nabla \varphi $$
$$ \nabla^2 \varphi = -4\pi q \delta(r-r') $$

we have

$$ p_{\text{field}} = \frac{1}{4\pi} \int dV \nabla \varphi \times \text{curl} A $$

choosing the gauge

$$ \text{div} A = 0 $$

we have

$$ p_{\text{field}} = \frac{q}{c} A $$

that's the physical meaning of the total momentum of a particle in a magnetic field

$$ p = mv + \frac{q}{c} A $$

of course, the Hamiltonian contains the kinetic energy

$$ E_{\text{kin}} = \frac{(mv)^2}{2m} = \frac{1}{2m} \left( p - \frac{q}{c} A \right)^2 $$

that's why it is usually written that

$$ p \rightarrow p - \frac{q}{c} A $$

or in operational representation

$$ \dot{H} = \frac{1}{2m} \left(-i\hbar \nabla - (e/c) A \right)^2 = \frac{i\hbar}{2mc} (\nabla A + A \nabla) + \frac{e^2}{2mc^2} A^2 $$
Magnetic moment

Magnetic moment of a closed loop carrying current $I$: \[ \mathbf{M}_i = \frac{I}{2c} \oint \mathbf{r} \times d\mathbf{l} = ISn \]

Magnetic field on the axis of a loop of radius $R$ at a distance $z$ is: \[ H_z = \frac{2M_i}{\left(R^2 + z^2\right)^{3/2}} \]

Total magnetic moment: \[ \mathbf{M} = \sum \mathbf{M}_i \] (superposition principle)
origin of magnetism (except for currents...)

\[ \mathbf{M}_{\text{ion}} = \gamma \hbar \mathbf{J} = -g \mu_B \mathbf{J} \]

\[ \hbar \mathbf{J} = \hbar \mathbf{L} + \hbar \mathbf{S} \]

\[ \gamma \text{ - gyromagnetic ratio} \]
\[ g \text{ - Landé factor} \]

\[ \mu_B = \frac{e \hbar}{2mc} \approx 9.274 \times 10^{-21} \text{ erg/G} \]

Bohr magneton

\[ g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \]

free electron:

\[ g = 2.0023 \approx 2.00 \]

Magnetic moment:

\[ M_e \approx \mu_B \]

\( J=S=1/2 \)
what about MPMS?

it measures a total magnetic moment in cgs (emu)

1 emu is:
- M of a 1 m$^2$ loop carrying a 1 mA current
- M of a loop of radius 1.78 cm carrying a 1 A current
- Typical permanent magnet (1 mm$^3$) ~ 1 emu
- M of a neutron star ~ $10^{30}$ emu
- The Earth’s magnetic moment ~ $8 \times 10^{25}$ emu
- An electron spin: $\mu_B$ ~ $10^{-20}$ emu
- Proton and neutron: $\mu_N$ ~ $10^{-23}$ emu
- One Abrikosov vortex (0.1 mm long) ~ $10^{-10}$ emu
- Change in M due to d-wave gap < $10^{-10}$ emu/K
- Hard superconductors ~ 0.1 emu
magnetic moment in a magnetic field

Energy: \[ W = -\mathbf{\mu B} = -\mathbf{\mu B} \cos(\theta) \]

Force: \[ F = -\text{grad}(W) = \text{grad}(\mathbf{\mu B}) \]

For example, for \[ \mathbf{B} = [B_x(x), 0, 0], \mathbf{\mu} = (\mu_x, \mu_y, 0) \]

\[ \mathbf{\mu B} = \mu_x B_x \text{ and } F = \mu_x \frac{dB_x}{dx} \]

In inhomogeneous magnetic field

\( \mu_x > 0 \)

\( \mu_x < 0 \)

\( F \) changes sign however torque aligns along the field
Practical definitions

per unit volume

$$4\pi M = B - H$$

bound (molecular, spin etc) currents
free currents (moving charge)

What is the problem with this definition?
1. Assumes uniform $H$ (ellipsoids only)
2. Assumes uniform $B$ (homogeneous system)

$$\chi = \frac{M}{H} \text{ - dimensionless!}$$

$$\chi_{SI} = 4\pi \chi_{cgs}$$

some other quantities are used:

$$\chi_m = \frac{\chi}{\rho} \left[ \frac{cc}{cc \cdot g} = g^{-1} \right]$$

$$\chi_{mol} = \chi_m M_m \left[ \frac{g}{g \cdot mol} = mol^{-1} \right]$$
Let’s see how well it works

a simple classical paramagnet

\[ M = M_s L(x), \quad x = \frac{p \mu_B H}{k_B T} \]

\[ \frac{\mu_B}{k_B} = 0.671 \left[ \frac{K}{T} \right] \]
Spatial inhomogeneity

M is not a single-valued function of H

demagnetization energy, magnetostatic energy, dipolar or fringe (stray) fields energy

\[ \sim M^2 V \]

ferromagnetic domains

domain walls energy

\[ \sigma \sim 1 \frac{erg}{cm^2} \]

gain – energy of field outside loose – energy of the domain wall
Magnetic hysteresis

Diagram showing the hysteresis loop with axes for magnetic field strength ($H$) and magnetic flux density ($B$). The loop illustrates the behavior of magnetic materials under varying magnetic fields, highlighting the irreversibility in boundary displacements.
type-II superconductor

Meissner State

Partial Penetration

Trapped Flux
<table>
<thead>
<tr>
<th>( \hat{H}(\omega) )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{J}(\omega) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \hat{E}(\omega) )</td>
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</tr>
<tr>
<td>( \hat{M}(\omega) )</td>
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<tr>
<td></td>
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<tr>
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<tr>
<td></td>
<td>-1</td>
</tr>
</tbody>
</table>
Using Faraday law of induction

\[ V = -\frac{1}{c} \frac{d\Phi}{dt} \]
How to measure magnetic field?

Miniature Hall-probe Array